A Primer on Directional Similarity

John Newell

Part 2: Similar Triangles & The Pythagorean Theorem

The Basic Math

We can begin with the Pythagorean theorem. Most people with a twentieth century education in mathematics know the Pythagorean theorem as an equation, namely: $a^2 + b^2 = c^2$, where *a* and *b* are the sides forming a right angle in a right triangle and *c* is the hypotenuse (which is the ancient Greek way of saying 'the side opposite the right angle'). Pythagoras, however, regarded this equation as a short hand way of describing the following picture:

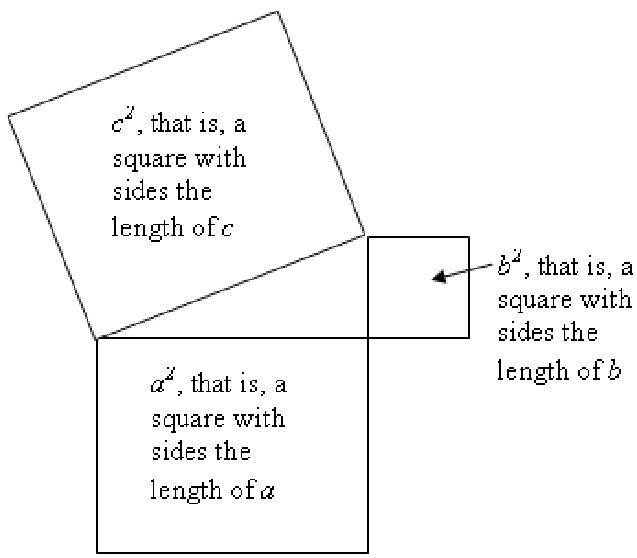

Figure 1: The Pythagorean Theorem

In other words, the big square (c^2) is equal in area to the areas of the other two squares *combined* (a^2+ b^2). In order to see how this turns out to be the case, we should take a minute to discuss similar triangles.

Similar Triangles

Similar triangles are triangles which are similar in shape, but different in size. One way to think of it is if you have two things with the same shape and one is close by, while the other is some distance away. The further one will appear smaller. To go a step further, you could take a photograph of an object up close, and then another from across the room. It is the same object in both pictures (so the shape is the same), but in one picture it is larger, and the other it is smaller. The *object* itself is the same size, but the *pictures* show a difference, so the images in the pictures are similar to one another (and to the object itself); they are not equal to one another because their sizes are different.

We can play the same game with triangles. With triangles (which are named after their three '*tri*' angles), similarity arises if the angle sizes are the same, while the lines that connect the angles become longer or shorter. One reason this occurs is that the angles control the ratio of the sizes of the sides—you can't stretch one side without stretching the other sides *if* you want to keep the angles the same size. If you draw a triangle and take pictures of it when it is close by and further away, you will see that the angles remain the same size, while the sides become larger or shorter *proportionately*—that is, the *relation* of their sizes (say one is twice as long as the other) remains the same, even if the size itself changes (10 to 20 has the same relation as 1 to 2 or 5 to 10—in traditional math, we write these expressions as 10:20, 1:2 and 5:10; in modern math, we write them as fractions 10/20, 1/2, and 5/10).

Another way of finding similar triangles is illustrated in the following figures. We are using right angles in these pictures because we are trying to move toward understanding the Pythagorean Theorem, but any kind of triangle could be used. The basic trick is to use parallel lines, because they preserve the angle sizes. In Figure 2 we have the right triangle *ABC* and the parallel line *DE* drawn within it. The right angles are marked with boxes, so they are equal, and the angle at A is the same for both triangle *ABC* and triangle *ADE*, so those angles are equal. That leaves angles *DEA* and *BCA*, which must be equal because the angles of a Euclidean triangle have to add up to 180°, so the angles are equal and the triangles are similar.

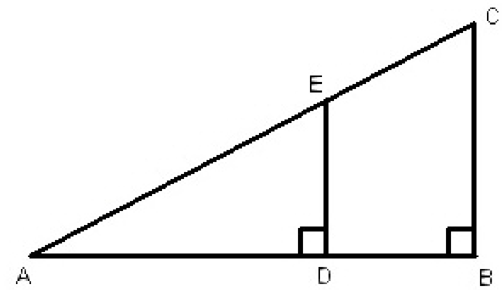

Figure 2: Similar Triangles *ABC* and *ADE*

The same story can be told about Figure 3, which draws the line in the triangle parallel to *AC* instead of *BC*. Here the common angle is at *C*, while the equal angles formed by the parallel lines are at *A* and *E*, as well as the right angles at *D* and *B*.

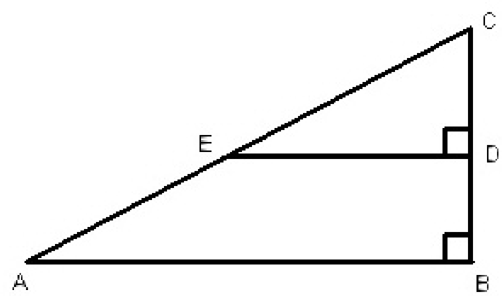

Figure 3: Similar Triangles *ABC* and *ADE*

Figure 4 adds a new twist by using both parallels, and isolating the similar triangle that occurs at their intersection.

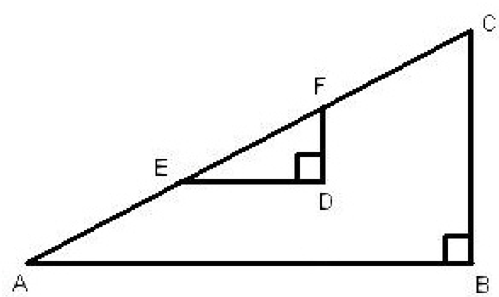

Figure 4: Similar Triangles *ABC* and *EDF*

Figure 5 adds a new twist by flipping things around and producing three similar triangles instead of two. To do this, we draw a perpendicular (not a parallel) from the right angle *B* to the hypotenuse, *AC*. In this case, we have two right angles as marked, a common angle at *A* for triangles *ABC* and *ADB* and so equal remaining angles at *ACB* and *ABD*, and so triangles *ABC* and *ADB* are similar.

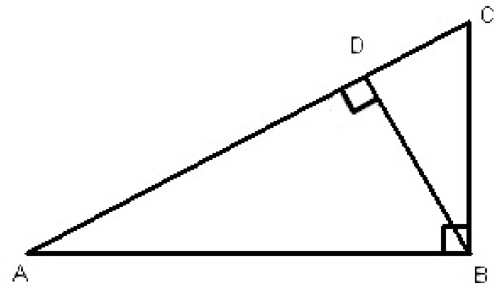

Figure 5: Similar triangles *ABC*, *ADB* and *BDC*

In Figure 5 we also have a shared angle at *C* between triangles *ABC* and *BDC*, as well as right angles at *B* and *D*, and so equal remaining angles *CBD* and *BAC*. Consequently, triangle (or △) *BDC* is similar to △*ABC*. But we saw that △*ABC* was similar to △*ADB* because their angles were equal, so the angles of △*BDC* are equal to the angles of △*ADB*, and so these two inner triangles are similar to one another.

With the triangles in Figures 2-4, it is easy to find the corresponding sides: they are either parallel or coincide. Thus, in Figure 2, *AD* corresponds to *AB*, *DE* corresponds to *BC*, and *AE* to *AC*. In Figure 5, the corresponding sides are flipped around, and there are three triangles to work with, so keeping things straight can take a little work. Table 1 identifies the corresponding parts for the three triangles.

	△*ABC*	**△*ADB***	**△*BDC***
Long leg of right angle	*AB*	*AD*	*BC*
Short leg of right angle	*BC*	*DB*	*CD*
Hypotenuse	*AC*	*AB*	*BC*

Table 1: Corresponding sides of triangles in Figure 5

The corresponding sides are important because these sides of the triangles are proportional, that is, the same ratio that obtains between sides *AB* and *AC* in △*ABC*, should obtain between the corresponding sides of △*ADB* and △*BDC*. In modern math, when things are in the same ratio, we can write this as an equation of fractions, for example: 10/20 = 5/10. In the traditional format, we would write 10 : 20 :: 5 : 10. The traditional format avoids the sign of equality because the traditional understanding of numbers is less generalized than the modern understanding of numbers. Thus, the moderns feel comfortable assigning a number to any length whatsoever, whereas the traditional method only allows assigning numbers that are commensurate with a given unit length—in their book, anything else is 'irrational,' that is, there is no ratio (*ir-* + *ratio*) between them.

One way to see the difference is to think about π, the moderns treat it as a number (3.14…), the ancients regard it as a relation (the ratio of the diameter of a circle to its circumference). The traditional way is rather easy to grasp: you can see the circle, its circumference is there to inspect, and its diameter is easy to find, all of these are finite tasks. With the modern approach, you deal with a 'number' which has a non-finite trail of decimal points—you can never actually say the number, but you can give it a nickname (π) and work with an approximation of it.

When Pythagoras was working on his theorem, he was thinking in terms of traditional ratios. Looking at Figure 5, he would have said that triangles *ABC* and *ADB* are proportional, that is, *AB* is to *AC* (in $\triangle ABC$) just as *AD* is to *AB* (in $\triangle ADB$), or, in traditional mathematical shorthand: *AB* : *AC* :: *AD* : *AB.*

Proportions can be manipulated just as fractions are, and the rule for cross-multiplication applies. Thus if we have *5/10 = 10/20*, we can cross multiply and come up with equal products; in this case *5 x 20 = 10 x 10*, which is to say that *100 = 100*. With our proportion *AB* : *AC* :: *AD* : *AB,* this means that $AB^2 = AC,AD$, and that, in turn, means that a square with a side the length of *AB* is equal in area to a rectangle with sides *AC* and *AD*. Figure 6 depicts this.

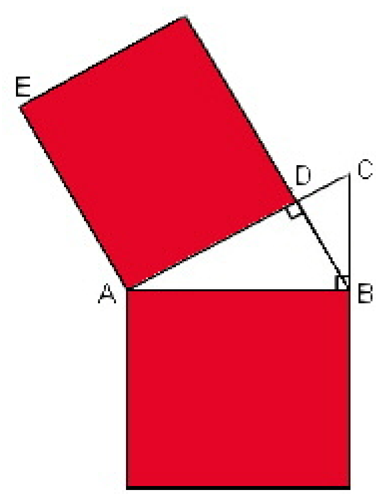

Figure 6: The square on *AB* equals the rectangle on *AD* with sides equal to *AC* (*AE* = *AC*).

Likewise, a telling proportion obtains between triangles *ABC* and *BDC*, namely: *BC* : *AC* :: *CD* : *BC*, and so $BC^2 = AC,CD$. Figure 7 combines this finding with that depicted in Figure 6.

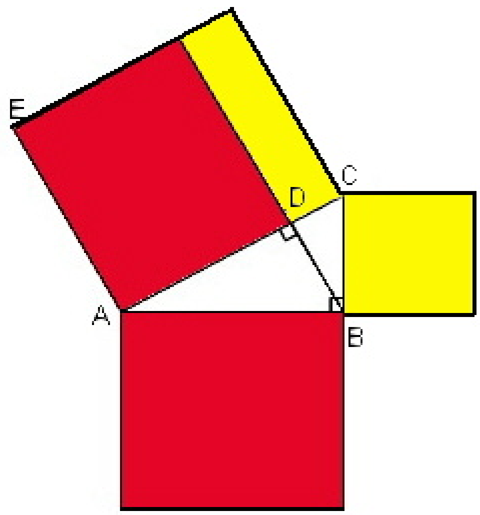

Figure 7: The square on *BC* equals the rectangle on *CD* with sides equal to *AC*, and so $AB^2 + BC^2 = AC^2$, because the square of *AC* equals the rectangle on *AD* (with side *AC*) plus the rectangle on *CD* (with side *AC*).

Thus we find rather quickly what Pythagoras discovered.